REPRESENTATION OF LEVEL PATHS OF AN ANALYTIC FUNCTION

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By a level path we mean a differentiable function (path function) which maps an interval on the real line into the complex plane, with the property that on its image, a given non-constant analytic function has constant modulus. So that the level path is a parameterization according to arc length, we require that its derivative have unit modulus everywhere on its interval of definition.

Proposition 1: If f is analytic on an open set $\Re$, and $z_0 \in \Re$ satisfies $f(z_0) \neq 0$ and $f'(z_0) \neq 0$, and for some interval $A \subseteq \mathbb{R}$ and some set $B \subseteq \{z \in \Re; f(z) f'(z) \neq 0\}$ there exists a differentiable function $p : A \to B$ satisfying

$$p'(s) = \frac{\left| f'(p(s)) \overline{f(p(s))} \right| i}{f'(p(s)) \overline{f(p(s))}}$$

and $p(s_0) = z_0$ for some $s_0 \in A$, then $|f(z)| = |f(z_0)|$ for all $z \in p(A)$. Further, p is a path function which parameterizes $p(A)$ according to arc length such that $p(s_0) = z_0$.

Proof: Suppose there exists $p : A \to B$ satisfying

$$p'(s) = \frac{\left| f'(p(s)) \overline{f(p(s))} \right| i}{f'(p(s)) \overline{f(p(s))}}$$

and $p(s_0) = z_0$ for some $s_0 \in A$. Note that for all $s \in A$,

$$\frac{d}{ds}\left(\left|f\left(p(s)\right)\right|^2\right)$$

$$=\frac{d}{ds}\left(f\left(p(s)\right)\overline{f\left(p(s)\right)}\right)=f'\left(p(s)\right)p'(s)\overline{f\left(p(s)\right)}+f\left(p(s)\right)\overline{f'\left(p(s)\right)}\overline{p'(s)}=0$$

Next, f is clearly continuous on $p(A)$, hence so is $|f|$, thus $|f(z)|$ is a (positive) constant on $p(A)$. Finally, note that $|p'(s)|=\left|\frac{\left|f'\left(p(s)\right)\overline{f\left(p(s)\right)}\right|i}{f'\left(p(s)\right)\overline{f\left(p(s)\right)}}\right|=1$, so p is a path which parameterizes $p(A)$ according to arc length such that $p(s_0)=z_0$, Q.E.D.[1] Thus there exists a path $p:A\rightarrow B$ such that $p(A)\subseteq\Re$ with $z_0\in p(A)$ such that $|f(z)|=|f(z_0)|$ for all $z\in p(A)$.

<u>Proposition 2</u>: If $g(z)$ exists such that it is analytic over $\{z\in\Re;\ f(z)\,f'(z)\neq 0\}$ and satisfies $f(z)=\exp g(z)$ there, then the level path of $f(z)$ satisfies

$$p'(s)=i\frac{\overline{g'\left(p(s)\right)}}{\left|g'\left(p(s)\right)\right|}.$$

<u>Proof</u>: If $f(z)=\exp g(z)$, then $f'(z)=g'(z)\,f(z)$, hence

$$p'(s)=\frac{\left|f'\left(p(s)\right)\overline{f\left(p(s)\right)}\right|i}{f'\left(p(s)\right)\overline{f\left(p(s)\right)}}=\frac{\left|g'\left(p(s)\right)f\left(p(s)\right)\overline{f\left(p(s)\right)}\right|i}{g'\left(p(s)\right)f\left(p(s)\right)\overline{f\left(p(s)\right)}}=i\frac{\overline{g'\left(p(s)\right)}}{\left|g'\left(p(s)\right)\right|}$$, Q.E.D.

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[1] The alternative definition $p'(s)=-\frac{\left|f'\left(p(s)\right)\overline{f\left(p(s)\right)}\right|i}{f'\left(p(s)\right)\overline{f\left(p(s)\right)}}$ would have sufficed, and corresponds to a parameterization of $p(A)$ in the opposite direction.

In the following we will write $c = \left| f\left(z_0\right) \right|$ for ease of notation, so that $\left| f\left(z\right) \right| = c$ for all $z \in p\left(A\right)$. Using this notation, we may write

$p'\left(s\right) = \dfrac{\left| f'\left(p\left(s\right)\right) \right| ci}{f'\left(p\left(s\right)\right) \overline{f\left(p\left(s\right)\right)}}$. After squaring both sides and some rearrangement we get that $f'\left(p\left(s\right)\right) \overline{f\left(p\left(s\right)\right)}^2 p'\left(s\right)^2 = -c^2 \overline{f'\left(p\left(s\right)\right)}$. Taking the derivative with respect to s yields

$$\begin{aligned} & f''\left(p\left(s\right)\right) p'\left(s\right) \overline{f\left(p\left(s\right)\right)}^2 p'\left(s\right)^2 + f'\left(p\left(s\right)\right) \overline{2 f\left(p\left(s\right)\right) f'\left(p\left(s\right)\right) p'\left(s\right)} p'\left(s\right)^2 \\ & + f'\left(p\left(s\right)\right) \overline{f\left(p\left(s\right)\right)}^2 2 p'\left(s\right) p''\left(s\right) = -c^2 \overline{f''\left(p\left(s\right)\right) p'\left(s\right)} \end{aligned}$$

which immediately yields

$$p''\left(s\right) = -\frac{c^2 \overline{f''\left(p\left(s\right)\right)} \overline{p'\left(s\right)}^2}{2 f'\left(p\left(s\right)\right) \overline{f\left(p\left(s\right)\right)}^2} - \frac{f''\left(p\left(s\right)\right) p'\left(s\right)^2}{2 f'\left(p\left(s\right)\right)} - \frac{\overline{f'\left(p\left(s\right)\right)}}{\overline{f\left(p\left(s\right)\right)}}$$

after application of the identity $p'\left(s\right) \overline{p'\left(s\right)} = 1$. Higher derivatives may be obtained iteratively by differentiating with respect to s and then making use of the identity $p'\left(s\right) \overline{p'\left(s\right)} = 1$. We then have the power series expansion

$$\begin{aligned} p\left(s\right) &= \sum_{n=0}^{\infty} \frac{p^{(n)}\left(s_0\right)}{n!} \left(s - s_0\right)^n = p\left(s_0\right) + p'\left(s_0\right)\left(s - s_0\right) \\ & -\frac{1}{2}\left(\frac{c^2 \overline{f''\left(p\left(s_0\right)\right)} \overline{p'\left(s_0\right)}^2}{2 f'\left(p\left(s_0\right)\right) \overline{f\left(p\left(s_0\right)\right)}^2} + \frac{f''\left(p\left(s_0\right)\right) p'\left(s_0\right)^2}{2 f'\left(p\left(s_0\right)\right)} + \frac{\overline{f'\left(p\left(s_0\right)\right)}}{\overline{f\left(p\left(s_0\right)\right)}} \right)\left(s - s_0\right)^2 + \cdots \\ &= z_0 + p'\left(s_0\right)\left(s - s_0\right) \\ & -\frac{1}{2}\left(\frac{c^2 \overline{f''\left(z_0\right)} \overline{p'\left(s_0\right)}^2}{2 f'\left(z_0\right) \overline{f\left(z_0\right)}^2} + \frac{f''\left(z_0\right) p'\left(s_0\right)^2}{2 f'\left(z_0\right)} + \frac{\overline{f'\left(z_0\right)}}{\overline{f\left(z_0\right)}} \right)\left(s - s_0\right)^2 + \cdots \end{aligned}$$

where

$$p'(s_0)=\frac{|f'(p(s_0))|ci}{f'(p(s_0))\overline{f(p(s_0))}}=\frac{|f'(z_0)|ci}{f'(z_0)\overline{f(z_0)}}.$$

Example: Let us consider the case $f(z)=az$, with $a\neq 0$ and $z_0\neq 0$. The corresponding power series about s_0 is

$$p(s)=z_0+\frac{z_0 i}{|z_0|}(s-s_0)-\frac{1}{2}\left(\frac{\bar{a}}{\overline{az_0}}\right)(s-s_0)^2+\cdots=z_0\left(1+\frac{i}{|z_0|}(s-s_0)-\frac{1}{2}\frac{1}{|z_0|^2}(s-s_0)^2+\cdots\right)$$

This is confirmed by the differential equation

$$p'(s)=\frac{\left|f'(p(s))\overline{f(p(s))}\right|i}{f'(p(s))\overline{f(p(s))}}$$

which in this case is

$$p'(s)=\frac{p(s)i}{|p(s)|}.$$

Under the conditions $p(s_0)=z_0$ and $|p'(s)|=1$ this differential equation has the unique solution

$$p(s)=z_0\exp\left(\frac{i(s-s_0)}{|z_0|}\right).$$

This agrees with the power series above.