A HOMOTOPY METHOD FOR FINDING EIGENVALUES AND EIGENVECTORS OF A LINEAR OPERATOR

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1 INTRODUCTION

Suppose we want to find the eigenvalues and eigenvectors for a linear operator L on a Hilbert space $\mathcal{H}$. Suppose also that L has a complete orthonormal set of eigenvectors. Finally, suppose that we have solved this problem for some other operator K on $\mathcal{H}$, with K also possessing a complete orthonormal set of eigenvectors. Is it possible to represent the eigenvalues and eigenvectors of L in terms of the corresponding properties of K? In this paper we address this question.

2 APPROACH

We use a homotopy technique. We form a convex combination $\theta L+(1-\theta)K$ of the operators K and L with parameter $\theta\in[0,1]$. As the parameter ranges from 0 to 1, the operator $\theta L+(1-\theta)K$ ranges from K to L. More precisely, at each value of θ the corresponding operator is given by $(\theta L+(1-\theta)K)(f)=\theta(L(f))+(1-\theta)(K(f))$

Next, we envision, for each choice of θ, a set of eigenvectors $\{f_i(\theta)\}_{i=1}^{\infty}$ and eigenvalues $\{\lambda_i(\theta)\}_{i=1}^{\infty}$ for the corresponding operator $\theta L+(1-\theta)K$, satisfying $\lambda_i(\theta)f_i(\theta)=\theta(L(f_i(\theta)))+(1-\theta)(K(f_i(\theta)))$. Then $f_i(0)=e_i$, where $\{e_i\}_{i=1}^{\infty}$ are the eigenvectors of K, and $\{\lambda_i(0)\}_{i=1}^{\infty}$ are the corresponding eigenvalues.

Our strategy is to develop power series representations of $\left\{ f_i(\theta) \right\}_{i=1}^{\infty}$ and $\left\{ \lambda_i(\theta) \right\}_{i=1}^{\infty}$, and then, if possible, to evaluate them at the value $\theta = 1$. If the power series representations converge there, we may expect to find that $\left\{ f_i(1) \right\}_{i=1}^{\infty}$ and $\left\{ \lambda_i(1) \right\}_{i=1}^{\infty}$ are the eigenvectors and eigenvalues of the operator L.

3 DERIVATION

We stipulated earlier that $\left\{ e_i \right\}_{i=1}^{\infty}$ are the eigenvectors of K, whence $f_i(0) = e_i$. By assumption, we may stipulate that they are orthonormal: $(e_m, e_n) = \delta_m^n$. Let $\left\{ \lambda_i(0) \right\}_{i=1}^{\infty}$ represent their corresponding eigenvalues, so that $K(e_i) = \lambda_i(0) e_i$. Taking the inner product we get $\left(K(e_i), e_j\right) = \left(\lambda_i(0) e_i, e_j\right) = \lambda_i(0)\delta_i^j$. In case $i = j$ we get

$$\lambda_i(0) = \left(K(e_i), e_i\right)$$

Next, we define $\left\{ \varphi_{i,j}(\theta) \right\}_{i,j=1}^{\infty}$ as $\varphi_{i,j}(\theta) = \left(f_i(\theta), e_j \right)$. We then have that

$f_i(\theta) = \sum_{j=1}^{\infty} \varphi_{i,j}(\theta) e_j$. In particular

$$\boxed{\varphi_{i,j}(0) = \delta_j^i}$$

Making the appropriate substitutions into

$\lambda_i(\theta) f_i(\theta) = \theta\left(L\left(f_i(\theta)\right)\right) + (1-\theta)\left(K\left(f_i(\theta)\right)\right),$

we get $\lambda_i(\theta) \sum_{j=1}^{\infty} \varphi_{i,j}(\theta) e_j = \theta\left(L\left(\sum_{j=1}^{\infty} \varphi_{i,j}(\theta) e_j \right)\right) + (1-\theta)\left(K\left(\sum_{j=1}^{\infty} \varphi_{i,j}(\theta) e_j \right)\right).$

After rearrangement we get

$$\lambda_i(\theta)\sum_{j=1}^{\infty}\varphi_{i,j}(\theta)e_j=\theta\sum_{j=1}^{\infty}\varphi_{i,j}(\theta)L(e_j)+(1-\theta)\sum_{j=1}^{\infty}\varphi_{i,j}(\theta)K(e_j)$$

and after taking the inner product with e_k there results the equation

$$\lambda_i(\theta)\sum_{j=1}^{\infty}\varphi_{i,j}(\theta)(e_j,e_k)=\theta\sum_{j=1}^{\infty}\varphi_{i,j}(\theta)(L(e_j),e_k)+(1-\theta)\sum_{j=1}^{\infty}\varphi_{i,j}(\theta)(K(e_j),e_k).$$

Recalling that $(e_i,e_j)=\delta_i^j$ and $(K(e_i),e_j)=\lambda_i(0)\delta_i^j$, and defining

$X_{i,j}\equiv(L(e_i),e_j)$ leads, after substitution, to the equation

$$\lambda_i(\theta)\sum_{j=1}^{\infty}\varphi_{i,j}(\theta)\delta_j^k=\theta\sum_{j=1}^{\infty}\varphi_{i,j}(\theta)X_{j,k}+(1-\theta)\sum_{j=1}^{\infty}\varphi_{i,j}(\theta)\lambda_j(0)\delta_j^k$$

Simplifying yields

$$\boxed{\lambda_i(\theta)\varphi_{i,k}(\theta)=\theta\sum_{j=1}^{\infty}X_{j,k}\varphi_{i,j}(\theta)+(1-\theta)\varphi_{i,k}(\theta)\lambda_k(0)}$$

The approach from this point is to calculate about $\theta=0$ the Maclaurin series expansions of the $\{f_i(\theta)\}_{i=1}^{\infty}$ and $\{\lambda_i(\theta)\}_{i=1}^{\infty}$. To do this our procedure is to solve for the coefficients of power series which formally solve the above equation. That is, we assume the forms $\lambda_i(\theta)=\sum_{n=0}^{\infty}a_{i,n}\theta^n$ and $\varphi_{i,j}(\theta)=\sum_{r=0}^{\infty}b_{i,j,r}\theta^r$ and substitute them in that equation, then solve for the coefficients $\{a_{i,n}\}$ and $\{b_{i,j,r}\}$.

After performing the substitution, we get

$\sum_{n=0}^{\infty}a_{i,n}\theta^n\sum_{m=0}^{\infty}b_{i,k,m}\theta^m=\theta\sum_{j=1}^{\infty}\sum_{r=0}^{\infty}X_{j,k}b_{i,j,r}\theta^r+(1-\theta)\sum_{r=0}^{\infty}\lambda_k(0)b_{i,k,r}\theta^r$. Upon rearranging, there results $\sum_{s=0}^{\infty}\theta^s\sum_{m=0}^{s}a_{i,s-m}b_{i,k,m}=\sum_{r=1}^{\infty}\theta^r\sum_{j=1}^{\infty}X_{j,k}b_{i,j,r-1}+\sum_{r=0}^{\infty}\lambda_k(0)b_{i,k,r}\theta^r+\sum_{r=1}^{\infty}-\lambda_k(0)b_{i,k,r-1}\theta^r$. We first set equal the coefficients of the constant terms to get $\sum_{m=0}^{0} a_{i,0-m} b_{i,k,m} = \lambda_k(0) b_{i,k,0}$,

which when simplified becomes $(a_{i,0} - \lambda_k(0)) b_{i,k,0} = 0$. Since $\lambda_i(\theta) = \sum_{n=0}^{\infty} a_{i,n}\theta^n$, it follows that $a_{i,0} = \lambda_i(0)$, hence $(\lambda_i(0) - \lambda_k(0)) b_{i,k,0} = 0$, thus $b_{i,k,0} = 0$ for $i \neq k$. Since $b_{i,i,0} = \varphi_{i,i}(0) = \delta_i^i = 1$, these conditions may be summarized by the single definition $b_{i,k,0} = \delta_i^k$. Next, for $r > 0$ we set equal the coefficients of the θ^r terms to get

$\sum_{m=0}^{r} a_{i,r-m} b_{i,k,m} = \sum_{j=1}^{\infty} X_{j,k} b_{i,j,r-1} + \lambda_k(0) b_{i,k,r} - \lambda_k(0) b_{i,k,r-1}$. Upon substituting $b_{i,k,0} = \delta_i^k$ we get $a_{i,r}\delta_i^k + \sum_{m=1}^{r} a_{i,r-m} b_{i,k,m} = \sum_{j=1}^{\infty} X_{j,k} b_{i,j,r-1} + \lambda_k(0) b_{i,k,r} - \lambda_k(0) b_{i,k,r-1}$. The examination of this equation splits naturally into the two cases $i = k$ and $i \neq k$. Considering the case $i = k$ first, we get $a_{i,r} = \sum_{j=1}^{\infty} X_{j,i} b_{i,j,r-1} + \lambda_i(0) b_{i,i,r} - \lambda_i(0) b_{i,i,r-1} - \sum_{m=1}^{r} a_{i,r-m} b_{i,i,m}$. In case $i \neq k$, we get $\sum_{m=1}^{r} a_{i,r-m} b_{i,k,m} = \sum_{j=1}^{\infty} X_{j,k} b_{i,j,r-1} + \lambda_k(0) b_{i,k,r} - \lambda_k(0) b_{i,k,r-1}$, which may be rearranged as $a_{i,0} b_{i,k,r} = \sum_{j=1}^{\infty} X_{j,k} b_{i,j,r-1} + \lambda_k(0)(b_{i,k,r} - b_{i,k,r-1}) - \sum_{m=1}^{r-1} a_{i,r-m} b_{i,k,m}$. After substituting $a_{i,0} = \lambda_i(0)$ and solving, we get

$$b_{i,k,r} = \frac{1-\delta_i^k}{\lambda_i(0) - \lambda_k(0)} \left(\sum_{j=1}^{\infty} X_{j,k} b_{i,j,r-1} - \sum_{m=1}^{r-1} a_{i,r-m} b_{i,k,m} - \lambda_k(0) b_{i,k,r-1} \right)$$

We now perform this recursion to get some initial results. Using

$a_{i,r} = \sum_{j=1}^{\infty} X_{j,i} b_{i,j,r-1} + \lambda_i(0) b_{i,i,r} - \lambda_i(0) b_{i,i,r-1} - \sum_{m=1}^{r} a_{i,r-m} b_{i,i,m}$, we see that

$a_{i,1} = \sum_{j=1}^{\infty} X_{j,i} b_{i,j,0} + \lambda_i(0) b_{i,i,1} - \lambda_i(0) b_{i,i,0} - \sum_{m=1}^{1} a_{i,1-m} b_{i,i,m}$ hence

$a_{i,1} = \sum_{j=1}^{\infty} X_{j,i} \delta_i^j + \lambda_i(0) b_{i,i,1} - \lambda_i(0) \delta_i^i - \lambda_i(0) b_{i,i,1}$ and thus $a_{i,1} = X_{i,i} - \lambda_i(0)$. Using

$b_{i,k,r} = \frac{1-\delta_i^k}{\lambda_i(0) - \lambda_k(0)} \left(\sum_{j=1}^{\infty} X_{j,k} b_{i,j,r-1} - \sum_{m=1}^{r-1} a_{i,r-m} b_{i,k,m} - \lambda_k(0) b_{i,k,r-1} \right)$, we get that

$b_{i,k,1} = \frac{1-\delta_i^k}{\lambda_i(0) - \lambda_k(0)} \left(\sum_{j=1}^{\infty} X_{j,k} b_{i,j,0} - \sum_{m=1}^{0} a_{i,r-m} b_{i,k,m} - \lambda_k(0) b_{i,k,0} \right)$ and thus

$b_{i,k,1} = \left(1-\delta_i^k\right) \frac{X_{i,k}}{\lambda_i(0) - \lambda_k(0)}$. Similarly we use the recursion formula to produce

$a_{i,2} = \sum_{j=1}^{\infty} X_{j,i} b_{i,j,1} + \lambda_i(0)\left(b_{i,i,2} - b_{i,i,1}\right) - \sum_{m=1}^{2} a_{i,2-m} b_{i,i,m}$, which after substitution yields

$a_{i,2} = \sum_{j=1}^{\infty} X_{j,i} \left(1-\delta_i^j\right) \frac{X_{i,j}}{\lambda_i(0) - \lambda_j(0)} = \sum_{\substack{j=1 \\ j\neq i}}^{\infty} \frac{X_{i,j} X_{j,i}}{\lambda_i(0) - \lambda_j(0)}$. Finally, we get

$b_{i,k,2} = \frac{1-\delta_i^k}{\lambda_i(0) - \lambda_k(0)} \left(\sum_{j=1}^{\infty} X_{j,k} b_{i,j,1} - \sum_{m=1}^{1} a_{i,2-m} b_{i,k,m} - \lambda_k(0) b_{i,k,1} \right)$ which after substitution

becomes $b_{i,k,2} = \frac{1-\delta_i^k}{\lambda_i(0) - \lambda_k(0)} \left(\sum_{\substack{j=1 \\ j\neq i}}^{\infty} X_{j,k} \frac{X_{i,j}}{\lambda_i(0) - \lambda_j(0)} + X_{i,k} \left(1 - \frac{X_{i,i}}{\lambda_i(0) - \lambda_k(0)} \right) \right)$. This

means that

$$\boxed{\lambda_i(\theta) = \lambda_i(0) + \left(X_{i,i} - \lambda_i(0)\right)\theta + \sum_{\substack{j=1 \\ j\neq i}}^{\infty} \frac{X_{i,j} X_{j,i}}{\lambda_i(0) - \lambda_j(0)} \theta^2 + \cdots}$$

and

$$f_i(\theta)=e_i+\sum_{\substack{j=1\\j\neq i}}^{\infty}\left(\frac{X_{i,j}}{\lambda_i(0)-\lambda_j(0)}\theta+\left(\begin{array}{l}\frac{1}{\lambda_i(0)-\lambda_j(0)}\sum_{\substack{k=1\\k\neq i}}^{\infty}\frac{X_{i,k}X_{k,j}}{\lambda_i(0)-\lambda_k(0)}\\+\frac{X_{i,j}}{\lambda_i(0)-\lambda_j(0)}-\frac{X_{i,i}X_{i,j}}{\left(\lambda_i(0)-\lambda_j(0)\right)^2}\end{array}\right)\theta^2+\cdots\right)e_j$$

(at least formally). Now if it happens that these formal power series actually converge absolutely at $\theta=1$, we then have

$$\lambda_i(1)=X_{i,i}+\sum_{\substack{j=1\\j\neq i}}^{\infty}\frac{X_{i,j}X_{j,i}}{\lambda_i(0)-\lambda_j(0)}+\cdots$$

and

$$f_i(1)=e_i+\sum_{\substack{j=1\\j\neq i}}^{\infty}\left(\frac{X_{i,j}}{\lambda_i(0)-\lambda_j(0)}+\left(\begin{array}{l}\frac{1}{\lambda_i(0)-\lambda_j(0)}\sum_{\substack{n=1\\n\neq i}}^{\infty}\frac{X_{i,n}X_{n,j}}{\lambda_i(0)-\lambda_n(0)}\\+\frac{X_{i,j}}{\lambda_i(0)-\lambda_j(0)}-\frac{X_{i,i}X_{i,j}}{\left(\lambda_i(0)-\lambda_j(0)\right)^2}\end{array}\right)+\cdots\right)e_j.$$

These are the eigenvalues and eigenvectors of the operator L, as desired.

4.1 EXAMPLE

Suppose we want to find the eigenvalues and eigenvectors for the linear operator L given by $L(f)(y)\equiv\int_{-\infty}^{\infty}G(x,y)\,f(x)\,dx$. That is, we want to find the functions $f:\mathbb{R}\to\mathbb{C}$ and the scalars $\lambda\in\mathbb{C}$ satisfying $L(f)=\lambda f$, i.e. $\lambda f(y)=\int_{-\infty}^{\infty}G(x,y)\,f(x)\,dx$. Also suppose that we have solved this problem for some other operator K whose form is given by $K(f)(y)=\int_{-\infty}^{\infty}F(x,y)\,f(x)\,dx$.

Suppose that $\{e_i(x)\}_{i=1}^{\infty}$ are the eigenvectors of K, whence $f_i(0,x)=e_i(x)$, and that they are orthonormal. Let $\{\lambda_i(0)\}_{i=1}^{\infty}$ represent their corresponding eigenvalues, so that $\lambda_i(0)e_i(y)=\int_{-\infty}^{\infty}F(x,y)e_i(x)dx=\int_{-\infty}^{\infty}F(x,y)f_i(0,x)dx$. Then

$$\boxed{\lambda_i(0)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(x,y)e_i(x)e_i(y)dxdy}$$

Next, define $\{\varphi_{i,j}(\theta)\}_{i,j=1}^{\infty}$ as $\varphi_{i,j}(\theta)=\int_{-\infty}^{\infty}f_i(\theta,x)e_j(x)dx$. Next we define

$X_{m,n}\equiv\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x,y)e_m(x)e_n(y)dxdy$. Applying the results derived above, we get that if it happens that the following formal power series actually converge at $\theta=1$, we then have

$$\boxed{\lambda_i(1)=X_{i,i}+\sum_{\substack{j=1\\ j\neq i}}^{\infty}\frac{X_{i,j}X_{j,i}}{\lambda_i(0)-\lambda_j(0)}+\cdots}$$

and

$$\boxed{\begin{aligned}&f_i(1,x)\\&=e_i(x)+\sum_{\substack{j=1\\ j\neq i}}^{\infty}\left(\frac{X_{i,j}}{\lambda_i(0)-\lambda_j(0)}+\left(\begin{aligned}&\frac{1}{\lambda_i(0)-\lambda_j(0)}\sum_{\substack{n=1\\ n\neq i}}^{\infty}\frac{X_{i,n}X_{n,j}}{\lambda_i(0)-\lambda_n(0)}\\&+\frac{X_{i,j}}{\lambda_i(0)-\lambda_j(0)}-\frac{X_{i,i}X_{i,j}}{\left(\lambda_i(0)-\lambda_j(0)\right)^2}\end{aligned}\right)+\cdots\right)e_j(x)\end{aligned}}.$$

These are the eigenvalues and eigenvectors of the operator

$L(f)(y)\equiv\int_{-\infty}^{\infty}G(x,y)f(x)dx$, as desired.

4.2 EXAMPLE

Suppose we want to find the eigenvalues and eigenvectors for the linear operator L given by $L(f) \equiv Gf$, where G is an $n \times n$ Hermitian matrix over $\mathbb{C}$. That is, we want to find the $n \times 1$ matrices f and the scalars $\lambda \in \mathbb{C}$ satisfying $L(f) = \lambda f$, i.e. $\lambda f = Gf$. Also suppose that we have solved this problem for some other $n \times n$ Hermitian matrix over $\mathbb{C}$, namely K.

Suppose that $\{e_i\}_{i=1}^{\infty}$ are the eigenvectors of K, whence $f_i(0) = e_i$. Since K is Hermitian, they are orthonormal. Let $\{\lambda_i(0)\}_{i=1}^{\infty}$ represent their corresponding eigenvalues, so that $\lambda_i(0) e_i = K e_i$. Then

$$\boxed{\lambda_i(0) = (K e_i) \cdot e_i}$$

Next, define $\{\varphi_{i,j}(\theta)\}_{i,j=1}^{\infty}$ as $\varphi_{i,j}(\theta) = f_i(\theta) \cdot e_j$. Next we define $X_{m,n} \equiv (G e_m) \cdot e_n$.

Applying the results derived above, we get that if it happens that the following formal power series actually converge at $\theta = 1$, we then have

$$\boxed{\lambda_i(1) = X_{i,i} + \sum_{\substack{j=1 \\ j \neq i}}^{\infty} \frac{X_{i,j} X_{j,i}}{\lambda_i(0) - \lambda_j(0)} + \cdots}$$

and

$$\boxed{f_i(1) = e_i + \sum_{\substack{j=1 \\ j \neq i}}^{\infty} \left(\frac{X_{i,j}}{\lambda_i(0) - \lambda_j(0)} + \left(\begin{array}{l} \dfrac{1}{\lambda_i(0) - \lambda_j(0)} \displaystyle\sum_{\substack{n=1 \\ n \neq i}}^{\infty} \frac{X_{i,n} X_{n,j}}{\lambda_i(0) - \lambda_n(0)} \\ + \dfrac{X_{i,j}}{\lambda_i(0) - \lambda_j(0)} - \dfrac{X_{i,i} X_{i,j}}{\left(\lambda_i(0) - \lambda_j(0)\right)^2} \end{array} \right) + \cdots \right) e_j}.$$

These are the eigenvalues and eigenvectors of the operator $L(f) \equiv Gf$, as desired.