SOLUTIONS OF BILINEAR EQUATIONS IN A BOOLEAN RING WITH IDENTITY

Kerry M. Soileau

May 12, 2005

ksoileau@yahoo.com

In a Boolean ring with identity B, given $\{a_1, a_2, \cdots, a_N, b\} \subseteq B$, we seek the conditions under which it is possible to solve the following equation in the unknowns $\{x_1, x_2, \cdots, x_N\} \subseteq B$:

$$\sum_{i=1}^{N} a_i x_i = b,$$

and to find at least one solution for every case in which a solution exists.

Proposition: A solution exists if and only if $b\prod_{n=1}^{N}(1+a_n) = 0$.

Proof: $(\Rightarrow)$ Suppose a solution $\{x_1, x_2, \cdots, x_N\}$ exists. Then

$$b\prod_{n=1}^{N}(1+a_n) = \sum_{i=1}^{N} a_i x_i \prod_{n=1}^{N}(1+a_n) = \sum_{i=1}^{N} x_i \left(a_i \prod_{n=1}^{N}(1+a_n) \right) = \sum_{i=1}^{N} x_i \cdot 0 = 0.$$

$(\Leftarrow)$ Conversely, suppose $b\prod_{n=1}^{N}(1+a_n) = 0$. Then

$$b = b + b\prod_{n=1}^{N}(1+a_n) = b + b + ba_1 + \sum_{i=2}^{N} a_i b \prod_{j=1}^{i-1}(1+a_j) = ba_1 + \sum_{i=2}^{N} a_i b \prod_{j=1}^{i-1}(1+a_j),$$

hence a solution $\{x_1, x_2, \cdots, x_N\} \subseteq B$ is given by $x_1 = b$, $x_i = b\prod_{j=1}^{i-1}(1+a_j)$ for $i = 2, 3, \cdots, N$.

Note: Given any solution set $\{x_1, x_2, \cdots, x_N\} \subseteq B$, it follows that for any $c \in B$, $\{(1+c+bc)x_1, (1+c+bc)x_2, \cdots, (1+c+bc)x_N\} \subseteq B$ is also a solution. We may use an alternative "vector" notation for this solution "space" as follows: $(x_1, x_2, \cdots, x_N) + c((1+b)x_1, (1+b)x_2, \cdots, (1+b)x_N) \subseteq B^N$.