STATE VECTOR DETERMINATION BY A SINGLE TRACKING SATELLITE

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ABSTRACT

Using only a single tracking satellite capable of only range measurements to an orbiting object in an unknown Keplerian orbit, it is theoretically possible to calculate the orbit and a current state vector. In this paper we derive the algorithm that can perform this calculation.

1. INTRODUCTION

The purpose of this paper is to explore the limits of usefulness of a single tracking satellite capable of only range measurement of another orbiting object. We show that using only a single tracking satellite capable of only range measurements to an orbiting object in an unknown Keplerian orbit, it is theoretically possible to calculate the orbit and a current state vector. These results might be of interest in situations where a constellation of many GPS-like satellites is infeasible, e.g. Mars orbit.

2. ASSUMPTIONS

Suppose we have a tracking satellite in circular orbit about some massive body (such as a planet or moon). We assume the satellite is capable of measuring range to another satellite in an unknown (but assumed Keplerian) orbit. We assume the masses of the satellites are insignificant in comparison to the mass of the body. We denote the time-varying position vectors of the tracking and tracked satellites by $\vec{R}$ and $\vec{r}$, respectively. To simplify the following derivation, we choose our time unit to be the time taken for the tracking satellite to travel one radian in its orbit, and our distance unit to be the radius of the tracking satellite's orbit. We may then write $\vec{R} = \hat{R}$ and $\vec{r} = r\hat{r}$, where $r = |\vec{r}|$.

In the following derivation, we will make use of the fact that

$$\frac{d\hat{R}}{dt} = \hat{T} \qquad \frac{d\hat{r}}{dt} = \dot{\theta}\hat{\tau} \qquad \frac{d\hat{T}}{dt} = -\hat{R} \qquad \frac{d\hat{\tau}}{dt} = -\dot{\theta}\hat{r}$$

where $\hat{T}$ and $\hat{\tau}$ are the velocity unit vectors for the tracking and tracked satellite, respectively, and where $\dot{\theta}$ is the angular velocity of the tracked satellite.

3. DERIVATION

First we note that if we define $q \equiv \left|\vec{r} - \vec{R}\right|^2$, then $q = \left(\vec{r} - \vec{R}\right)\cdot\left(\vec{r} - \vec{R}\right) = r^2 - 2\vec{r}\cdot\vec{R} + 1$.

Next, we introduce the parameter $s \equiv \ln r$. Accordingly, we have $q = e^{2s} - 2e^s \hat{r}\cdot\hat{R} + 1$.

Next, since the tracked satellite is in a Keplerian orbit we have that $r^2\dot{\theta} = C$ for some positive constant C. Replacing r with e^s we get $e^{2s}\dot{\theta} = C$, hence $\dot{\theta} = Ce^{-2s}$ and $\frac{d\hat{r}}{dt} = Ce^{-2s}\hat{\tau}$. We also note that $\frac{d\hat{\tau}}{dt} = -\dot{\theta}\hat{r} = -Ce^{-2s}\hat{r}$.

In the polar formulation of the equation $\ddot{\vec{r}} = -\frac{1}{r^2}\hat{r}$, we note the relation

$\ddot{r} = r\dot{\theta}^2 - \frac{1}{r^2} = e^s\left(Ce^{-2s}\right)^2 - \frac{1}{\left(e^s\right)^2} = C^2e^{-3s} - e^{-2s}$. Because of the identity $r = e^s$, after two successive differentiations with respect to time we get $\ddot{r} = \ddot{s}e^s + \dot{s}^2e^s$. Equating the two expressions for $\ddot{r}$ yields $C^2e^{-3s} - e^{-2s} = \ddot{s}e^s + \dot{s}^2e^s$. Solving, we obtain $\ddot{s} = C^2e^{-4s} - e^{-3s} - \dot{s}^2$. Rearranging, we have immediately that $C^2 = e^s + \left(\ddot{s} + \dot{s}^2\right)e^{4s}$.

Differentiating with respect to time yields $0 = \dot{s}e^s + \left(\dddot{s} + 2\dot{s}\,\ddot{s}\right)e^{4s} + \left(\ddot{s} + \dot{s}^2\right)4\dot{s}e^{4s}$, whence

$e^s = \left(\frac{-\dot{s}}{\dddot{s} + \left(6\ddot{s} + 4\dot{s}^2\right)\dot{s}}\right)^{\frac{1}{3}}$. Substituting into the equation $C^2 = e^s + \left(\ddot{s} + \dot{s}^2\right)e^{4s}$, we get

$$C = \left(\left(\ddot{s} + \dot{s}^2 \right) \left(\frac{\dot{s}}{\dddot{s} + 6\dot{s}\ddot{s} + 4\dot{s}^3} \right)^4 - \frac{\dot{s}}{\dddot{s} + 6\dot{s}\ddot{s} + 4\dot{s}^3} \right)^{\frac{1}{2}}.$$

Repeated differentiation of $q = 1 + e^{2s} - 2e^s \hat{r} \cdot \hat{R}$ with respect to time, together with the following identities

$$\frac{d\hat{r}}{dt} = Ce^{-2s}\hat{\tau} \qquad \frac{d\hat{\tau}}{dt} = -Ce^{-2s}\hat{r} \qquad \frac{d\hat{R}}{dt} = \hat{T} \qquad \frac{d\hat{T}}{dt} = -\hat{R} \qquad \ddot{s} = C^2 e^{-4s} - e^{-3s} - \dot{s}^2$$

yields

$$q = 1 + e^{2s} - 2e^s \hat{r} \cdot \hat{R}$$

$$\dot{q} = 2\dot{s}e^{2s} - 2\dot{s}e^s \hat{r} \cdot \hat{R} - 2e^s \hat{r} \cdot \hat{T} - 2Ce^{-s} \hat{\tau} \cdot \hat{R}$$

$$\ddot{q} = 2C^2 e^{-2s} - 2e^{-s} + 2\dot{s}^2 e^{2s} + \left(2e^{-2s} + 2e^s \right) \hat{r} \cdot \hat{R} - 4\dot{s}e^s \hat{r} \cdot \hat{T} - 4Ce^{-s} \hat{\tau} \cdot \hat{T}$$

$$\dddot{q} = -2\dot{s}e^{-s} + \left(-4\dot{s}e^{-2s} + 6\dot{s}e^s \right) \hat{r} \cdot \hat{R} + \left(6e^{-2s} + 2e^s \right) \hat{r} \cdot \hat{T} + \left(2Ce^{-4s} + 6Ce^{-s} \right) \hat{\tau} \cdot \hat{R}$$

These in turn can be used to derive the following equations:

$$\hat{r} \cdot \hat{R} = \tfrac{1}{2} e^s + \tfrac{1}{2} e^{-s} - \tfrac{1}{2} q e^{-s}$$

$$\hat{r} \cdot \hat{T} = \frac{\left(\tfrac{3}{4}\dot{s} - \tfrac{3}{4} q\dot{s} + \tfrac{1}{4}\dot{q} \right) e^{-4s} + \tfrac{3}{4}\dot{s}e^{-2s} + \left(\tfrac{1}{4}\dddot{q} + \tfrac{3}{4}\dot{q} \right) e^{-s} - \tfrac{3}{2}\dot{s}e^s}{e^{-3s} - 1}$$

$$\hat{\tau} \cdot \hat{R} = \frac{\left(-\tfrac{5}{4C}\dot{s} + \tfrac{5}{4C} q\dot{s} - \tfrac{3}{4C}\dot{q} \right) e^{-2s} - \tfrac{1}{4C}\dot{s} + \left(-\tfrac{1}{4C}\dddot{q} - \tfrac{1}{4C}\dot{q} + \tfrac{1}{2C}\dot{s} - \tfrac{1}{2C} q\dot{s} \right) e^s + \tfrac{1}{C}\dot{s}e^{3s}}{e^{-3s} - 1}$$

$$\hat{\tau} \cdot \hat{T} = \frac{1}{e^{-3s} - 1} \begin{pmatrix} \tfrac{1}{4C}(1-q)e^{-5s} + \tfrac{1}{2}Ce^{-4s} - \tfrac{1}{4C}e^{-3s} + \left(\tfrac{3}{4C} q\dot{s}^2 - \tfrac{1}{4C}\dot{q}\dot{s} - \tfrac{3}{4C}\dot{s}^2 - \tfrac{1}{4C}\ddot{q} \right) e^{-2s} - \tfrac{1}{2}Ce^{-s} \\ -\tfrac{1}{4C}\dot{s}^2 + \tfrac{1}{2C} + \left(-\tfrac{1}{4C}\dddot{q}\,\dot{s} - \tfrac{3}{4C}\dot{q}\dot{s} - \tfrac{1}{4C} + \tfrac{1}{4C} q + \tfrac{1}{4C}\ddot{q} \right) e^s + \left(-\tfrac{1}{4C} + \tfrac{1}{C}\dot{s}^2 \right) e^{3s} \end{pmatrix}$$

We may now use the earlier equations to compute the values of $\hat{r} \cdot \hat{R}$, $\hat{r} \cdot \hat{T}$, $\hat{\tau} \cdot \hat{R}$, and $\hat{\tau} \cdot \hat{T}$.

Finally, we compute the quantities $\hat{r} \cdot \hat{Z}$ and $\hat{\tau} \cdot \hat{Z}$ where $\hat{Z} \equiv \hat{R} \times \hat{T}$.

$$\hat{r} = \left(\hat{r}\cdot\hat{R}\right)\hat{R} + \left(\hat{r}\cdot\hat{T}\right)\hat{T} + \left(\hat{r}\cdot\hat{Z}\right)\hat{Z}$$
$$\hat{\tau} = \left(\hat{\tau}\cdot\hat{R}\right)\hat{R} + \left(\hat{\tau}\cdot\hat{T}\right)\hat{T} + \left(\hat{\tau}\cdot\hat{Z}\right)\hat{Z}$$
$$1 = \left(\hat{r}\cdot\hat{R}\right)^2 + \left(\hat{r}\cdot\hat{T}\right)^2 + \left(\hat{r}\cdot\hat{Z}\right)^2$$
$$1 = \left(\hat{\tau}\cdot\hat{R}\right)^2 + \left(\hat{\tau}\cdot\hat{T}\right)^2 + \left(\hat{\tau}\cdot\hat{Z}\right)^2$$
$$\left|\hat{r}\cdot\hat{Z}\right| = \sqrt{1-\left(\hat{r}\cdot\hat{R}\right)^2 - \left(\hat{r}\cdot\hat{T}\right)^2}$$
$$\left|\hat{\tau}\cdot\hat{Z}\right| = \sqrt{1-\left(\hat{\tau}\cdot\hat{R}\right)^2 - \left(\hat{\tau}\cdot\hat{T}\right)^2}$$
$$0 = \hat{r}\cdot\hat{\tau} = \left(\hat{r}\cdot\hat{R}\hat{R} + \hat{r}\cdot\hat{T}\hat{T} + \hat{r}\cdot\hat{Z}\hat{Z}\right)\cdot\left(\hat{\tau}\cdot\hat{R}\hat{R} + \hat{\tau}\cdot\hat{T}\hat{T} + \hat{\tau}\cdot\hat{Z}\hat{Z}\right)$$
$$= \hat{r}\cdot\hat{R}\hat{\tau}\cdot\hat{R} + \hat{r}\cdot\hat{T}\hat{\tau}\cdot\hat{T} + \hat{r}\cdot\hat{Z}\hat{\tau}\cdot\hat{Z}$$
$$\hat{r}\cdot\hat{Z}\hat{\tau}\cdot\hat{Z} = -\hat{r}\cdot\hat{R}\hat{\tau}\cdot\hat{R} - \hat{r}\cdot\hat{T}\hat{\tau}\cdot\hat{T}$$

There can be at most two possibilities for the pair $\left(\hat{r}\cdot\hat{Z}, \hat{\tau}\cdot\hat{Z}\right)$. This is because the

equation $\left|\hat{r}\cdot\hat{Z}\right| = \sqrt{1-\left(\hat{r}\cdot\hat{R}\right)^2 - \left(\hat{r}\cdot\hat{T}\right)^2}$ implies that

$\hat{r}\cdot\hat{Z} = \pm\sqrt{1-\left(\hat{r}\cdot\hat{R}\right)^2 - \left(\hat{r}\cdot\hat{T}\right)^2}$, and then the equations

$$\left|\hat{\tau}\cdot\hat{Z}\right| = \sqrt{1-\left(\hat{\tau}\cdot\hat{R}\right)^2 - \left(\hat{\tau}\cdot\hat{T}\right)^2}$$

and

$\hat{r}\cdot\hat{Z}\hat{\tau}\cdot\hat{Z} = -\hat{r}\cdot\hat{R}\hat{\tau}\cdot\hat{R} - \hat{r}\cdot\hat{T}\hat{\tau}\cdot\hat{T}$ determine the value of $\hat{\tau}\cdot\hat{Z}$.

The state vector of the tracked satellite corresponding to the solution e^s is then given by

$$\vec{r} = e^s\left(\hat{r}\cdot\hat{R}\right)\hat{R} + e^s\left(\hat{r}\cdot\hat{T}\right)\hat{T} + e^s\left(\hat{r}\cdot\hat{Z}\right)\hat{Z}$$

and

$$\vec{v} = e^{-\frac{s}{2}}\left(\hat{\tau}\cdot\hat{R}\right)\hat{R} + e^{-\frac{s}{2}}\left(\hat{\tau}\cdot\hat{T}\right)\hat{T} + e^{-\frac{s}{2}}\left(\hat{\tau}\cdot\hat{Z}\right)\hat{Z}.$$

4. APPLICATION

The foregoing analysis shows that a single measurement of $\{q, \dot{q}, \ddot{q}, \dddot{q}\}$ and $\{\dot{s}, \ddot{s}, \dddot{s}\}$ will narrow the possibilities for the tracked state vector down to only two. By repeating this analysis using new measurements, we can eventually find the unique orbit consistent with all the measurements performed.

These results show what is possible given exact values of $\{q, \dot{q}, \ddot{q}, \dddot{q}\}$ and $\{\dot{s}, \ddot{s}, \dddot{s}\}$. In practice, exact values will not be available. There will be some error in measurement of q and $\dot{s}$. The values of q, $\dot{q}$, $\ddot{q}$ and $\dddot{q}$ may have to be estimated using well-known difference calculus techniques. Similarly, estimation of $\dot{s}$, $\ddot{s}$ and $\dddot{s}$ will probably be required.

5. CONCLUSION

Using only a single tracking satellite capable of only range measurements to an orbiting object in an unknown orbit, together with radial velocity measurements, it is theoretically possible to calculate the orbit and a current state vector. In this paper we derived the algorithm that can perform this calculation.

Appendix. CIRCULAR ORBIT

Differentiating

$\dddot{q} = -2\dot{s}e^{-s} + \left(-4\dot{s}e^{-2s} + 6\dot{s}e^{s}\right)\hat{r}\cdot\hat{R} + \left(6e^{-2s} + 2e^{s}\right)\hat{r}\cdot\hat{T} + \left(2Ce^{-4s} + 6Ce^{-s}\right)\hat{\tau}\cdot\hat{R}$ with respect to time yields

$$q^{(iv)} = \frac{1}{e^{-3s} - 1}\begin{pmatrix} \left(-3C^2 + 3C^2 q\right)e^{-10s} + \left(4 - 4q\right)e^{-9s} - C^2 e^{-8s} + \left(2 + 3C^2 - 3C^2 q\right)e^{-7s} \\ + \left(\frac{9}{2}\dot{s}^2 - \frac{3}{2C}\dot{s}^2 - 6 + 6q - 3q\dot{s}^2 - \frac{1}{2C}\dot{q}\dot{s} + \frac{7}{2}\dot{q}\dot{s} - \frac{1}{2C}\ddot{q} - \frac{3}{2}\ddot{q}\right)e^{-6s} + 5C^2 e^{-5s} \\ + \left(-\dot{s}^2 - 8\right)e^{-4s} + \left(3 - 12\dot{s}^2 + 12q\dot{s}^2 - 3q - 15\dot{q}\dot{s} - 3\dddot{q}\dot{s}\right)e^{-3s} - 4C^2 e^{-2s} \\ + \left(14\dot{s}^2 + 7\right)e^{-s} - 1 - 3\dot{s}^2 + 3\dot{s}^2 + 3q\dot{s}^2 + q - 3q\dot{s}^2 + 2\ddot{q} + \left(-4\dot{s}^2 - 1\right)e^{2s} \end{pmatrix}$$

In this special circular orbit case, $\dot{s}$ is constantly zero, so we have $0 = C^2 e^{-4s} - e^{-3s} - 0^2$

hence $C^2 = e^s$. Substitution in the equation above and some rearrangement yields

$$e^{11s} + \left(-q + 1 - 2\ddot{q} - q^{(iv)}\right)e^{9s} - 3e^{8s} + \left(q^{(iv)} - 3 + 3q\right)e^{6s} + 3e^{5s} + \left(-3q + 3 + 2\ddot{q}\right)e^{3s} - e^{2s}$$
$$-1 + q = 0$$

This can be factored as follows:

$$e^{11s} + \left(-q + 1 - 2\ddot{q} - q^{(iv)}\right)e^{9s} - 3e^{8s} + \left(q^{(iv)} - 3 + 3q\right)e^{6s} + 3e^{5s} + \left(-3q + 3 + 2\ddot{q}\right)e^{3s} - e^{2s} - 1 + q$$
$$= \left(e^{8s} + \left(-q^{(iv)} - q - 2\ddot{q} + 1\right)e^{6s} - 2e^{5s} + \left(2q - 2 - 2\ddot{q}\right)e^{3s} + e^{2s} + 1 - q\right)\left(e^{3s} - 1\right)$$

So if the tracked satellite's orbital altitude is not exactly equal to 1, we must have

$e^{8s} + \left(-q^{(iv)} - q - 2\ddot{q} + 1\right)e^{6s} - 2e^{5s} + \left(2q - 2 - 2\ddot{q}\right)e^{3s} + e^{2s} + 1 - q = 0$. Since $r \equiv e^s$, this

equation may be expressed as

$r^8 + \left(-q^{(iv)} - q - 2\ddot{q} + 1\right)r^6 - 2r^5 + \left(2q - 2 - 2\ddot{q}\right)r^3 + r^2 + 1 - q = 0$.